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An elliptic analogue of the multiple gamma function

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Abstract
A hierarchy of functions including the elliptic gamma function is introduced. It can be interpreted as an elliptic analogue of the multiple gamma function and its trigonometric limit coincides with a q-analogue of the multiple gamma function. Some properties of the functions are considered.

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1. Introduction

In various studies on mathematical physics, the elliptic functions play very important roles. Especially, since the discovery of the elliptic solution of the Yang–Baxter equation [3], many investigations of solvable lattice models and of commuting difference operators with elliptic coefficients have been explored.

Recently, these facts have been studied from the point of view of an 'elliptic analogue of special function theory'. Frenkel and Turaev [5] reformulate a relation of the elliptic $6j$-symbol as a well poised hypergeometric series of elliptic coefficients. Their idea was developed by Warnaar [27] and Rosengren [17]. They derived certain identities of elliptic hypergeometric series. Spiridonov and Zhedanov [21, 22] considered an orthogonal function with elliptic coefficients.

Among these results, one of the most important would be construction of the elliptic gamma function by Ruijsenaars [18]. He showed the existence and uniqueness of the elliptic gamma function $G_1(z|(\tau_0, \tau_1))$ which satisfies the functional relation

$$G_1(z + \tau_1|(\tau_0, \tau_1)) = \theta_0(z; \tau_0)G_1(z|(\tau_0, \tau_1))$$

where $\theta_0(z; \tau) := \prod_{k=0}^{\infty} \left(1 - e^{2\pi\sqrt{-1}(z+k\tau)}\right)\left(1 - e^{2\pi\sqrt{-1}(-z+(k+1)\tau)}\right)$. After his work, Zabrodin [28] considered Baxter's Q-operator by using this function. Felder and Varchenko [4] used it for their studies on the elliptic Kniznik–Zamolodchikov (KZ) equation and gave a cohomological interpretation of this function. Spiridonov [20] constructed an elliptic analogue of the beta integral. We should mention the elliptic Macdonald–Morris conjecture presented by van Diejen and Spiridonov [25].

In this paper, we construct a hierarchy of functions including the elliptic gamma function. They satisfy relations

$$G_r\left(z+\tau_r|\left(\tau_0,\ldots,\tau_r\right)\right)=G_{r-1}\left(z|\left(\tau_0,\ldots,\tau_{r-1}\right)\right)G_r\left(z|\left(\tau_0,\ldots,\tau_r\right)\right)$$
$$G_0\left(z|\left(\tau_0\right)\right):=\theta_0(z;\tau)$$

which can be considered as an elliptic analogue of the multiple gamma function. We call this function the *multiple elliptic gamma function*. The multiple gamma function was introduced by Barnes [2]. Vignéras [26] interpreted its special case as the function satisfying the generalized Bohr–Mollerup theorem. The multiple elliptic gamma function can also be characterized by using the defining relation and an initial value.

This paper is organized as follows: in section 2, we define a certain q-function and study its properties. It is applied to construct the multiple elliptic gamma function. In section 3, we construct the multiple elliptic gamma function and consider its uniqueness, some elementary properties and a kind of trigonometric limit.

Finally, we note that generalizations of the gamma function have been applied to construction of solutions for difference systems relevant to quantum integrable systems. For example, Jimbo and Miwa [10], Miwa and Takeyama [12] and Miwa *et al* [13] constructed integral solutions for the q-KZ equation by using a multiple sine function, which can be considered as a kind of trigonometric analogue of the multiple gamma function. The author and Ueno [15, 16] constructed an integral solution for hypergeometric q-difference systems. Ruijsenaars [19] used his generalized function in order to study an eigenfunction of commuting difference systems. The multiple elliptic gamma function is also expected to be applicable to the theory of elliptic special functions.

2. Multiple $\underline{q}$-shifted factorial

2.1. Definition and properties

First, we fix some notations. For $1 < j < r$, τ_j are complex parameters satisfying $\Im\tau_j > 0$. We put $q_j := \mathrm{e}^{2\pi\sqrt{-1}\tau_j}$, $x := \mathrm{e}^{2\pi\sqrt{-1}z}$, $\underline{\tau} := (\tau_0, \tau_1, \ldots, \tau_r)$ and $\underline{q} := (q_0, q_1, \ldots, q_r)$. For the set $\underline{\tau}$, we define $\underline{\tau}^+(j)$, $\underline{\tau}^-(j)$ and $|\underline{\tau}|$ as

$$\underline{\tau}^+(j) := (\tau_0, \tau_1, \ldots, \tau_r, \tau_j)$$
$$\underline{\tau}^-(j) := (\tau_0, \tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_r)$$
$$\underline{\tau}[j] := (\tau_0, \tau_1, \ldots, \tau_{j-1}, -\tau_j, \tau_{j+1}, \tau_r)$$
$$\left|\underline{\tau}\right| := \sum_{j=0}^{r}\tau_j.$$

Similarly, we introduce the following notation:

$$\underline{q}^-(j) := (q_0, q_1, \ldots, q_{j-1}, q_{j+1}, \ldots, q_r)$$
$$\underline{q}[j] := (q_0, q_1, \ldots, q_{j-1}, q_j^{-1}, q_{j+1}, \ldots, q_r).$$

We define the *multiple $\underline{q}$-shifted factorial* as the following infinite product (see Appel [1]):

Definition 2.1.

$$\left(x;\underline{q}\right)_\infty^{(r)} := \prod_{i_1,i_2,\ldots,i_r=0}^{\infty}\left(1 - xq_0^{i_0}q_1^{i_1}\cdots q_r^{i_r}\right).$$

This function is a meromorphic function of z and has its zeros at

$$z = m_0\tau_0 + m_1\tau_1 + \cdots + m_r\tau_r + n \qquad (m_j \in \boldsymbol{Z}_{\leqslant 0}, n \in \boldsymbol{Z})$$

when $\tau_j \neq \tau_k$ for $j \neq k$ and $0 \leqslant j, k \leqslant r$. $\left(x; \underline{q}\right)_\infty^{(r)}$ satisfies a functional relation

$$\left(q_j x; \underline{q}\right)_\infty^{(r)} := \frac{\left(x; \underline{q}\right)_\infty^{(r)}}{\left(x; \underline{q}^-(j)\right)_\infty^{(r-1)}}.$$

We can extend $\left(x; \underline{q}\right)_\infty^{(r)}$ in the sense of Felder and Varchenko [4]. If $\Im\tau_{j_0} < 0$ only for $j_0 \in \{0, \ldots, r\}$, then we can determine a unique function satisfying the above relation as

$$\left(x; \underline{q}\right)_\infty^{(r)} := \frac{1}{\left(q_{j_0}^{-1} x; \underline{q}[j_0]\right)_\infty^{(r)}}. \tag{1}$$

However, the statement about the positions of zeros and poles is not valid. In the following argument, we suppose $\Im\tau_j > 0$ for all $j \geqslant 0$ when the condition on $\underline{\tau}$ is not mentioned especially.

Next, we represent $\left(x; \underline{q}\right)_\infty^{(r)}$ as a product of other multiple $\underline{q}$-shifted factorials. For example, we consider the case when $r = 1$. We can separate the product as follows:

$$\begin{aligned}\left(x; \underline{q}\right)_\infty^{(1)} &= \prod_{i_0 \leqslant i_1} (1 - x q_0^{i_0} q_1^{i_1}) \prod_{i_1 < i_0} (1 - x q_0^{i_0} q_1^{i_1}) \\ &= \prod_{j_0, j_1} (1 - x q_0^{j_0} q_1^{j_0 + j_1}) \prod_{j_0, j_1} (1 - x q_0^{j_0 + j_1 + 1} q_1^{j_1}) \\ &= \left(x; (q_0 q_1, q_1)\right)_\infty^{(1)} \left(q_0 x; (q_0 q_1, q_0)\right)_\infty^{(1)}.\end{aligned}$$

Similarly, in the case $r = 2$, we have

$$\begin{aligned}\left(x; \underline{q}\right)_\infty^{(2)} = \left(x; (q_0 q_1 q_2, q_1 q_2, q_2)\right)_\infty^{(2)} &\left(x q_1; (q_0 q_1 q_2, q_1 q_2, q_1)\right)_\infty^{(2)} \\ \times \left(x q_0 q_2; (q_0 q_1 q_2, q_0 q_2, q_2)\right)_\infty^{(2)} &\left(x q_0; (q_0 q_1 q_2, q_0 q_2, q_0)\right)_\infty^{(2)} \\ \times \left(x q_0 q_1; (q_0 q_1 q_2, q_0 q_1, q_1)\right)_\infty^{(2)} &\left(x q_0^2 q_1; (q_0 q_1 q_2, q_0 q_1, q_0)\right)_\infty^{(2)}.\end{aligned}$$

We can generalize the above argument.

Proposition 2.2.

$$\left(x; \underline{q}\right)_\infty^{(r)} := \prod_\sigma \left(x \prod_{j=0}^{r-1} q_{\sigma(j)}^{d_{\sigma,j}}; \underline{p}^\sigma\right)_\infty^{(r)}$$

where σ runs over all permutations of $\{0, 1, \ldots, r\}$,

$$\underline{p}^\sigma := (p_0^\sigma, p_1^\sigma, \ldots, p_r^\sigma) \qquad \textit{where} \quad p_j^\sigma := \prod_{k=j}^{r} q_{\sigma(k)}$$

and $d_{\sigma,j}$ counts the number of adjacent inversions (cf [23])

$$d_{\sigma,j} := \begin{cases} 0 & \textit{if} \quad j = 0 \\ \#\left\{k \in \{0, \ldots, j\} \mid \sigma(k-1) \rangle \sigma(k)\right\} & \textit{if} \quad j > 0. \end{cases}$$

We remark a relation between $\left(x;\underline{q}\right)^{(r)}_{\infty}$ and a certain q-series. We define generalized q-polylogarithms as

$$\mathrm{Li}_{r+1}(x;(q_0,\ldots,q_{r-1})) := \sum_{j=1}^{\infty} \frac{x^j}{j\prod_{i=0}^{r-1}(1-q_i^j)} \qquad \text{for} \quad |x|<1.$$

This function can be defined by using Jackson integrals, as

$$\int_0^a f(t)\,\mathrm{d}_q t = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n$$

(cf [7]). $\mathrm{Li}_r(x;\underline{q})$ has the following iterated integral representation:

$$\mathrm{Li}_1(x) = -\log(1-x) \qquad \mathrm{Li}_2(x;q_0) = \frac{1}{1-q_0}\int_0^x \mathrm{Li}_1(x)\frac{\mathrm{d}_{q_0}t}{t}$$

$$\mathrm{Li}_{r+1}(x;(q_0,\ldots,q_{r-1})) = \frac{1}{1-q_{r-1}}\int_0^x \mathrm{Li}_r(t;(q_0,\ldots,q_{r-2}))\frac{\mathrm{d}_{q_{r-1}}t}{t} \qquad \text{for} \quad r\geqslant 2.$$

Then we have the following proposition:

Proposition 2.3.

$$(x;\underline{q})^{(r)}_{\infty} = \exp\left(-\mathrm{Li}_{r+2}(x;\underline{q})\right)$$

when $|x|<1$.

Proof. We take such a branch of logarithm that

$$\log(1-x) = \sum_{k=1}^{\infty}\frac{x^k}{k} \qquad \text{for} \quad |x|<1.$$

Through straightforward calculation, we have

$$\begin{aligned}\log\left(x;\underline{q}\right)^{(r)}_{\infty} &= \sum_{i_0,i_1,\ldots,i_r=0}^{\infty}\log(1-xq_0^{i_0}q_1^{i_1}\cdots q_r^{i_r})\\ &= -\sum_{j=1}^{\infty}\sum_{i_0,i_1,\ldots,i_r=0}^{\infty}\frac{1}{j}x^j q_0^{i_0 j}q_1^{i_1 j}\cdots q_r^{i_r j}\\ &= -\sum_{j=1}^{\infty}\frac{x^j}{j\prod_{i=0}^{r}(1-q_i^j)}.\end{aligned}$$

□

In the case $\tau_0=\tau_1=\cdots=\tau_r=\tau$, we can see that

$$\left(x;q\right)^{(r)}_{\infty} := \Bigg(x;\underbrace{(q,q,\ldots,q)}_{r+1}\Bigg)^{(r)}_{\infty} = \prod_{k=0}^{\infty}(1-xq^k)^{\binom{k+r-1}{r}}.$$

Proposition 2.3 is rewritten as follows:

$$(x;q)^{(r)}_{\infty} = \exp\left(-\mathrm{Li}_{r+2}(x;q)\right)$$

where $\mathrm{Li}_r(x;q)$ is Kirillov's q-polylogarithm [11]

$$\mathrm{Li}_r(x;q) := \sum_{j=1}^{\infty}\frac{x^j}{j(1-q^j)^{r-1}}.$$

2.2. Differential relation

We define $\chi_r\left(z|\,\underline{\tau}\right)$ as the logarithmic derivative of $\left(x;\,\underline{\tau}\right)^{(r)}_{\infty}$

$$\chi_r\left(z|\,\underline{\tau}\right) := \frac{\partial}{\partial z}\log\left(x;\,\underline{q}\right)^{(r)}_{\infty} = -\sum_{i_0,\ldots,i_r}^{\infty}\frac{2\pi\sqrt{-1}xq_0^{i_0}\cdots q_r^{i_r}}{1-xq_0^{i_0}\cdots q_r^{i_0}}.$$

Then, we can see that

$$\begin{aligned}\frac{\partial}{\partial\tau_j}\log\left(x;\,\underline{q}\right)^{(r)}_{\infty} &= -\sum_{i_0,\ldots,i_r=0}^{\infty}\frac{2\pi\sqrt{-1}i_jxq_0^{i_0}\cdots q_r^{i_r}}{1-xq_0^{i_0}\cdots q_r^{i_r}}\\ &= -\sum_{i_0,\ldots,i_r,i_{r+1}=0}^{\infty}\frac{2\pi\sqrt{-1}xq_0^{i_0}\cdots q_j^{i_j+i_{r+1}}\cdots q_r^{i_r}}{1-xq_0^{i_0}\cdots q_j^{i_j+i_{r+1}}\cdots q_r^{i_r}}\\ &\quad+\sum_{i_0,\ldots,i_r=0}^{\infty}\frac{2\pi\sqrt{-1}xq_0^{i_0}\cdots q_r^{i_r}}{1-xq_0^{i_0}\cdots q_r^{i_r}}\\ &= \chi_{r+1}\left(z|\,\underline{\tau}^{+}(j)\right)-\chi_r\left(z|\,\underline{\tau}\right).\end{aligned}$$

Thus, we have a differential relation between the logarithmic derivatives:

$$\frac{\partial}{\partial\tau_j}\chi_r(z|\underline{\tau}) = \frac{\partial}{\partial z}\chi_{r+1}\left(z|\,\underline{\tau}^{+}(j)\right)-\frac{\partial}{\partial z}\chi_r\left(z|\,\underline{\tau}\right) \tag{2}$$

$$= \frac{\partial}{\partial z}\chi_{r+1}\left(z+\tau_j\,\middle|\,\underline{\tau}^{+}(j)\right). \tag{3}$$

3. Multiple elliptic gamma function

3.1. Definition

In this section, we introduce the multiple gamma function $G_r(z|\underline{\tau})$.

Definition 3.1.

$$G_r\left(x|\,\underline{\tau}\right) := (x^{-1}q_0q_1\cdots q_r;\,\underline{q})^{(r)}_{\infty}\left\{(x;\,\underline{q})^{(r)}_{\infty}\right\}^{(-1)^r}.$$

We note that $G_1\left(x|\,(\tau_0,\tau_1)\right)$ is the elliptic gamma function [4,18]. In the case where $\forall\Im\tau_j>0$, this function is meromorphic and we can easily see the locations of zeros and poles of this function. For example, when $\tau_j\neq\tau_k$ for $j\neq k$, $j,k\in\{0,\ldots,r\}$, $G_r\left(x|\,\underline{\tau}\right)$ has the following zeros and poles: if r is odd, then $G_r(z|\underline{\tau})$ has simple zeros at

$$z = m_0\tau_0+m_1\tau_1+\cdots+m_r\tau_r+n \qquad (m_j\in\boldsymbol{Z}_{>0},\,n\in\boldsymbol{Z})$$

and simple poles at

$$z = m_0\tau_0+m_1\tau_1+\cdots+m_r\tau_r+n \qquad (m_j\in\boldsymbol{Z}_{\leqslant 0},\,n\in\boldsymbol{Z}).$$

If r is even, then $G_r(z|\underline{\tau})$ has simple zeros at

$$z = m_0\tau_0+m_1\tau_1+\cdots+m_r\tau_r+n \qquad (m_j\in\boldsymbol{Z},\,n\in\boldsymbol{Z})$$

and no poles.

We can see that $G_r\left(z|\,\underline{\tau}\right) = G_r\left(z|\,\underline{\tilde{\tau}}\right)$, where $\underline{\tilde{\tau}}$ is a permutation of the order of $\underline{\tau}$. From the definition of the multiple elliptic gamma function, we can see a functional relation of $G_r\left(z|\,\underline{\tau}\right)$

Proposition 3.2. (i) $G_r\left(z|\,\underline{\tau}\right)$ *satisfies a relation*

$$\begin{aligned}&G_r\left(z+1|\,\underline{\tau}\right)=G_r\left(z|\,\underline{\tau}\right)\\&G_r\left(z+\tau_j\middle|\,\underline{\tau}\right)=G_{r-1}\left(z|\,\underline{\tau}(j)\right)G_r\left(z|\,\underline{\tau}\right)\qquad \textit{for}\quad j=0,\ldots,r\\&G_0\left(z|\,(\tau_0)\right)=\theta_0(z;q_0)\end{aligned}$$

where

$$\theta_0(z;q_0):=(x;(q_0))^{(0)}_\infty(qx^{-1};(q_0))^{(0)}_\infty.$$

(ii) *At the point* $z=\left|\underline{\tau}\right|/2$, $G_r\left(z|\,\underline{\tau}\right)$ *takes the following value:*

$$G_r\left(\frac{|\underline{\tau}|}{2}\middle|\,\underline{\tau}\right)=\begin{cases}\left\{\left(q_0^{\frac12}q_1^{\frac12}\cdots q_r^{\frac12};\underline{q}\right)^{(r)}_\infty\right\}^2 & r\text{: even}\\ 1 & r\text{: odd.}\end{cases}$$

From relation (1), we can expand this function in a similar way to Felder and Varchenko [4]. If $\Im\tau_{j_0}<0$ for only one $j_0\in\{0,\ldots,r\}$, then

$$G_r\left(z|\,\underline{\tau}\right):=\frac{1}{G_r\left(z-\tau_{j_0}\middle|\,\underline{\tau}[j_0]\right)}\tag{4}$$

is a unique meromorphic function satisfying the first and the second relations of (i). However, the positions of zeros and poles are valid.

3.2. Uniqueness

We can prove a kind of uniqueness of the function satisfying the relation in proposition 3.2.

Proposition 3.3. *If* $G_{r-1}\left(z|\,(\tau_0,\ldots,\tau_{r-1})\right)$ $(\Im\tau_j>0)$ *is given, we can determine the unique meromorphic function* $u(z)$ *which satisfies:*

(i) $u(z)$ *is holomorphic upper half plane*;

(ii) $u(z+1)=u(z)\qquad u(z+\tau_r)=G_{r-1}\left(z|\,\underline{\tau}\right)u(z)$;

(iii) $u(\left|\underline{\tau}\right|/2)=\begin{cases}\left\{\left(q_0^{\frac12}q_1^{\frac12}\cdots q_r^{\frac12};\underline{q}\right)^{(r)}_\infty\right\}^2 & r\text{: even}\\ 1 & r\text{: odd.}\end{cases}$

Proof. This proposition can be proved by using the same argument as Felder and Varchenko [4]. For some j and some $\delta\in\boldsymbol{R}$, there exists a strip $0<\Im z<\Im\tau_j+\delta$ in which $u(z)=G_r\left(z|\,\underline{\tau}\right)$ has no zero. If $v(z)$ is another meromorphic function satisfying the above (i) $\sim$ (iii), then $v(z)/u(z)$ is doubly periodic and holomorphic on the strip. Thus, it is a bounded entire function. From Liouville's theorem, $v(z)/u(z)$ is a constant. From the condition (iii), it follows that $v(z)=u(z)$. □

Imposing the another condition on the set of the parameters, we can determine $G_r\left(z|\,\underline{\tau}\right)$ without the above condition (i).

Proposition 3.4. *For* $\underline{\tau}=(\tau_0,\tau_1,\ldots,\tau_r)$, *if there exist such* j *and* k *that* 1, τ_j *and* τ_k *are linearly independent over* $\boldsymbol{Q}$, *then we can determine a unique meromorphic function* $u(z)$ *satisfying*

$$\begin{aligned}&u(z+1)=u(z)\\&u(z+\tau_j)=G_{r-1}(z|\underline{\tau}(j))u(z)\qquad u(z+\tau_k)=G_{r-1}(z|\underline{\tau}(k))u(z)\\&u\left(\frac{|\underline{\tau}|}{2}\right)=\begin{cases}\left\{\left(q_0^{\frac12}q_1^{\frac12}\cdots q_r^{\frac12};\underline{q}\right)^{(r)}_\infty\right\}^2 & r\text{: even}\\ 1 & r\text{: odd.}\end{cases}\end{aligned}$$

Proof. We suppose that there exist two functions $u(z)$ and $v(z)$ satisfying the above relation. Then, from the relation

$$\frac{u(z+1)}{v(z+1)} = \frac{u(z+\tau_j)}{v(z+\tau_j)} = \frac{u(z+\tau_k)}{v(z+\tau_k)}$$

it follows that $u(z)/v(z)$ is a triple periodic meromorphic function, that is, constant. Therefore, $u(z)$ is equal to $v(z)$ because they take the same value at $z = \left|\underline{\tau}\right|/2$. □

From these propositions, it follows that for given $\underline{\tau}$ satisfying the above condition we determine $G_r\left(z|\,\underline{\tau}\right)$ if $G_{r-1}(z|(\tau_0,\ldots,\tau_{r-1}))$ is given.

3.3. Elementary properties

From the definition of the multiple elliptic gamma function, we derive some formulae by using straightforward calculation.

Proposition 3.5.

$$\text{(i)}\quad G_r\left(z|\,\underline{\tau}\right)\left\{G_r\left(\left|\underline{\tau}\right|-z|\,\underline{\tau}\right)\right\}^{(-1)^{r-1}} = 1$$

$$\text{(ii)}\quad G_r\left(z\left|\left(\frac{\tau_0}{N},\ldots,\frac{\tau_r}{N}\right)\right.\right) = \prod_{n_0,n_1,\ldots,n_r=0}^{N-1} G_r\left(z+\frac{n_0\tau_0+\cdots+n_r\tau_r}{N}\right|\underline{\tau}\Bigg).$$

Claim (i) is an analogue of the complementary formula of Euler's gamma function and claim (ii) is an analogue of the Gauss–Legendre multiplication formula for the gamma function.

Next, we represent $G_r\left(z|\,\underline{\tau}\right)$ by using trigonometric series. These are a generalization of the so-called 'summation formula' of Ruijsenaars [18] and Felder and Varchenko [4].

Proposition 3.6 (Summation formula). *If z is contained in the region*

$$z \in \left\{ z \in \boldsymbol{C} \,\middle|\, |\Im(2z-\left|\underline{\tau}\right|)| < \sum |\Im\tau_j| \right\}$$

then $G_r\left(z|\,\underline{\tau}\right)$ is represented by the following formula:

$$G_r\left(z|\,\underline{\tau}\right) = \begin{cases} \exp\left(\dfrac{1}{(2\sqrt{-1})^r}\displaystyle\sum_{l=1}^{\infty}\frac{\sin(\pi l(2z-\left|\underline{\tau}\right|))}{l\prod_{j=1}^{r}\sin\pi l\tau_j}\right) & r\text{: odd} \\ \exp\left(\dfrac{1}{2^r(\sqrt{-1})^{r+1}}\displaystyle\sum_{l=1}^{\infty}\frac{\cos(\pi l(2z-\left|\underline{\tau}\right|))}{l\prod_{j=1}^{r}\sin\pi l\tau_j}\right) & r\text{: even.} \end{cases}$$

We note that this formula is valid for $G_r\left(z|\,\underline{\tau}\right)$ with some dm $\tau_j < 0$ defined by using the relation (4).

From proposition 2.2, the next proposition follows.

Proposition 3.7.

$$G_r\left(z|\,\underline{\tau}\right) = \prod_{\sigma} G_r\left(z+\sum_{j=0}^{r-1} d_{\sigma,j}\tau_{\sigma(j)}\,\middle|\,\underline{\mu}^{\sigma}\right)$$

where σ is a permutation of $\{0,\ldots,r\}$, $d_{\sigma,j}$ was introduced in proposition 2.2 and

$$\underline{\mu}^{\sigma} := (\mu_0^{\sigma},\mu_1^{\sigma},\ldots,\mu_r^{\sigma}) \qquad \textit{where} \quad \mu_j^{\sigma} = \sum_{k=j}^{r}\tau_{\sigma(k)}.$$

3.4. Differential relation

We define $\psi_r(z|\underline{\tau})$ as the logarithmic derivative of $G_r\left(z|\,\underline{\tau}\right)$.

$$\psi_r(z|\underline{\tau}) := \frac{\mathrm{d}}{\mathrm{d}z}\log G_r\left(z|\,\underline{\tau}\right) = (-1)^r\chi_r(z|\underline{\tau}) + \chi_r(|\underline{\tau}| - z|\underline{\tau}).$$

Similarly to (2) and (3) we can derive a relation

$$\begin{aligned}\frac{\partial}{\partial\tau_j}\chi_r\left(|\underline{\tau}| - z\right|\underline{\tau}\big) &= -\frac{\partial}{\partial z}\chi_{r+1}\left(|\underline{\tau}^+(j)| - z\right|\underline{\tau}^+(j)\big) - \frac{\partial}{\partial z}\chi_r\left(|\underline{\tau}| - z\right|\underline{\tau}\big)\\ &= -\frac{\partial}{\partial z}\chi_{r+1}\left(\left|\underline{\tau}^+(j)\right| - (z+\tau_j)\right|\underline{\tau}^+(j)\big).\end{aligned}$$

Thus, we can see the following proposition:

Proposition 3.8. *There is a differential relation between* $\psi_r(z|\underline{\tau})$

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\tau_j}\right)\psi_r(z|\tau) = -\frac{\partial}{\partial z}\psi_{r+1}(z|\underline{\tau}^+(j))$$

or, equivalently,

$$\frac{\partial}{\partial\tau_j}\psi_r\left(z|\,\underline{\tau}\right) = -\frac{\partial}{\partial z}\psi_{r+1}\left(z+\tau_j\right|\underline{\tau}^+(j)\big).$$

Furthermore the multiple elliptic gamma function satisfies a differential relation

$$\left(\frac{\partial}{\partial\tau_j}G_r\left(z|\,\underline{\tau}\right)\right)G_{r+1}\left(z|\,\underline{\tau}^+(j)\right) + \frac{\partial}{\partial z}\left\{G_{r+1}\left(z+\tau_j\right|\underline{\tau}^+(j)\big)\right\} = 0.$$

3.5. Trigonometric limit

The multiple q-gamma function $G_r(z;q)$ was introduced as

$$G_0(z;a) := [z] := \frac{1-q^z}{1-q}$$

$$G_r(z+1;q) := (1-q)^{-\binom{z}{n}}\prod_{k=1}^{\infty}\left\{\left(\frac{1-q^{z+k}}{1-q^k}\right)^{\binom{-k}{n-1}}(1-q^k)^{g_r(z,k)}\right\} \qquad \text{for} \quad r \geqslant 1$$

where

$$g_r(z,u) = \binom{z-u}{n-1} - \binom{-u}{n-1}.$$

$\{G_r(z;q)\}_{r\geqslant 0}$ is a hierarchy of meromorphic functions satisfying a q-analogue of the generalized Bohr–Mollerup theorem

(i) $G_r(z+1;q) = G_{r-1}(z;q)G_r(z;q)$ (ii) $G_r(1;q) = 1$

(iii) $\dfrac{d^{r+1}}{dz^{r+1}}\log G_{r+1}(z+1;q) \geqslant 0$ for $z \geqslant 0$ (iv) $G_0(z;q) = [z]$

when $0 < q < 1$. We can relate our multiple elliptic gamma function to the multiple q-gamma function through a kind of trigonometric limit.

Proposition 3.9. *In the case when* $\tau_1 = \tau_2 = \cdots = \tau_r = \tau$*, as* $\tau_0/\sqrt{-1} \to \infty$*,*

$$G_r(\tau z|\underline{\tau})\prod_{k=0}^{r}\left(\left(q;(q_0,\underbrace{q,q,\ldots,q}_{k})\right)_\infty^{(r)}\right)^{\binom{z-1}{r-k}} \to G_r(z;q)$$

for z in any compact set in the domain $\boldsymbol{C}\backslash(\boldsymbol{Z}+\boldsymbol{Z}\tau)$ *where* $q := \mathrm{e}^{2\pi\sqrt{-1}\tau}$.

3.6. Cauchy determinant represented by the double elliptic gamma function

Vardi [24] remarked that a special case of the Cauchy determinant can be represented by using the double gamma function.

$$\det\left[\frac{1}{\alpha+i+j}\right]_{1\leqslant i,j\leqslant n} = G_2(n+1)^2\frac{G_2(n+2+\alpha)^2}{G_2(2+\alpha)G_2(2n+2+\alpha)} \tag{5}$$

where $G_2(z)$ is Barnes' G-function [2].

We can generalize this formula to the elliptic case. An elliptic analogue of the Cauchy determinant [6] is known as

$$\det\left[\frac{\theta(u+a_i+b_j)}{\theta(a_i+b_j)\theta(u)}\right]_{1\leqslant i,j\leqslant n} = \frac{\theta(u+\sum_{r=1}^n(a_r+b_j))}{\theta(u)}\frac{\prod_{1\leqslant i,j\leqslant n}\theta(a_j-a_i)\theta(b_j-b_i)}{\prod_{i,j=1,\dots,n}\theta(a_i+b_j)}$$

where $\theta(z)=\theta(z;\tau_0)$ is Jacobi's first theta function defined as

$$\theta(z)=\theta(z;\tau_0) := \sqrt{-1}\mathrm{e}^{\pi\sqrt{-1}(\tau_0/4-z)}\big(q_0;(q_0)\big)_\infty^{(0)}\theta_0(z;\tau_0).$$

In the case where

$$a_i = \mathrm{i}\tau_1+\beta \qquad b_j=\mathrm{j}\tau_2$$

we can represent this determinant by the following formula:

$$\det\left[\frac{\theta_0(u+\beta+\mathrm{i}\tau_1+\mathrm{j}\tau_2;\tau_0)}{\theta_0(\mathrm{i}\tau_1+\mathrm{j}\tau_2;\tau_0)(\theta_0(u;\tau_0))}\right]_{1\leqslant i,j,\leqslant n}$$

$$= \exp\left(\frac{n(n^2-1)\pi\sqrt{-1}(\tau_1+\tau_2)}{3}+\frac{n(n-1)}{2}\beta\right)\frac{\theta_0\left(u+n\beta+\frac{n(n+1)(\tau_1+\tau_2)}{2};\tau_0\right)}{\theta_0(u;\tau_0)}$$

$$\times\frac{G_2\left((n+1)\tau_1|\,(\tau_0,\tau_1,\tau_1)\right)G_2\left((n+1)\tau_2|\,(\tau_0,\tau_2,\tau_2)\right)}{G_1\left(\tau_1|\,(\tau_0,\tau_1)\right)^n G_1\left(\tau_2|\,(\tau_0,\tau_2)\right)^n G_2\left(\tau_1|\,(\tau_0,\tau_1,\tau_1)\right)G_2\left(\tau_2|\,(\tau_0,\tau_2,\tau_2)\right)}$$

$$\times\frac{G_2\left(\beta+(n+1)\tau_1+\tau_2|\,(\tau_0,\tau_1,\tau_2)\right)G_2\left(\beta+\tau_1+(n+1)\tau_2|\,(\tau_0,\tau_1.\tau,2)\right)}{G_2\left(\beta+(n+1)\tau_1+(n+1)\tau_2|\,(\tau_0,\tau_1,\tau_2)\right)G_2\left(\beta+\tau_1+\tau_2|\,(\tau_0.\tau_1,\tau_2)\right)}. \tag{6}$$

We take a trigonometric limit of (6) similarly to the argument in section 3.4. We put $\tau_1=\tau_2=\tau$ and $\beta=\tau\alpha$. As $\tau_0/\sqrt{-1}\to\infty$ and $u/\sqrt{-1}\to\infty$, we have the following determinant formula:

$$\det\left[\frac{1}{1-q^{\alpha+i+j}}\right]_{1\leqslant i,j\leqslant n} = q^{\frac{n(n^2-1)}{3}}\frac{G_2(1+n;q)^2G_2(\alpha+n+2;q)^2}{G_2(\alpha+2;q)G_2(\alpha+2(n+1);q)} \tag{7}$$

where $G_2(z;q)$ is the multiple q-gamma function [14]. This is a q-analogue of formula (5). We can obtain the formula (5) as the classical limit of the formula (7).

Hasegawa [8, 9] derived a generalization of the elliptic Cauchy determinant:

$$\det\left[\prod_{r=1}^n\theta(a_i+b_j+hY_{r<i}+(u-(\mathrm{i}-1)h)\delta_{r,i})\right]_{1\leqslant i,j\leqslant}$$

$$=\theta\left(u+\sum_{r=1}^n(a_i+b_j)\right)\prod_{i=1}^{n-1}\theta(u-\mathrm{i}h)\prod_{1\leqslant i<j\leqslant n}\theta(a_j-a_i)\theta(b_j-b_i)$$

where $\delta_{i,j}$ is Kronecker's delta and $Y_{i\cdot j}$ is defined as follows:

$$Y_{i,j} := \begin{cases}1 & \text{if}\quad i<j\\ 0 & \text{otherwise.}\end{cases}$$

In the case when

$$a_i = \mathrm{i}\tau_1 \qquad b_j = \mathrm{j}\tau_2$$

we can represent a special case of Hasegawa's formula by using the double elliptic gamma function:

$$\det\left[\prod_{r=1}^{n}\theta_0(\mathrm{i}\tau_1+\mathrm{j}\tau_2+hY_{r<i}+(u-(\mathrm{i}-1)h)\delta_{r,i};\tau_0)\right]_{1\leqslant i,j\leqslant n}$$
$$= \exp\left(\frac{n(n^2-1)\pi\sqrt{-1}}{3}(\tau_1+\tau_2)\right)\frac{G_1\left(u|\left(\tau_0,h\right)\right)}{G_1\left(u-(n-1)h|\left(\tau_0,h\right)\right)}$$
$$\times\frac{G_2\left((n+1)\tau_1|\left(\tau_0,\tau_1,\tau_1\right)\right)G_2\left((n+1)\tau_2|\left(\tau_0,\tau_2,\tau_2\right)\right)}{G_1\left(\tau_1|\left(\tau_0,\tau_1\right)\right)^n G_1\left(\tau_2|\left(\tau_0,\tau_2\right)\right)^n}.$$

The trigonometric limit and rational limit of this formula is represented by the following formula:

$$\det\left[\prod_{r=1}^{n}\left(1-q^{\mathrm{i}\tau_1+\mathrm{j}\tau_2+hY_{r<i}+(u-(\mathrm{i}-1)h)\delta_{r,i}}\right)\right]_{1\leqslant i,j\leqslant n} = q^{\frac{n(n^2-1)}{3}}\frac{G_1(u;q)G_2(n+1;q)^2}{G_1(u-(n-1)h;q)}$$

$$\det\left[\prod_{r=1}^{n}\left(\mathrm{i}\tau_1+\mathrm{j}\tau_2+hY_{r<i}+(u-(\mathrm{i}-1)h)\delta_{r,i}\right)\right]_{1\leqslant i,j\leqslant n} = \frac{G_1(u)G_2(n+1)^2}{G_1(u-(n-1)h)}.$$

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